International Journal of Heat and Mass Transfer 172 (2021) 121084

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/hmt

Unsteady convective heat transfer from a flat plate with heat flux that varies in space and time

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A R T I C L E I N F O

Article history:
Received 4 November 2020
Revised 19 January 2021
Accepted 6 February 2021

Keywords:
External convective heat transfer
Unsteady convective heat transfer
Integral energy equation
Karman-Pohlhausen polynomials
Hyperbolic partial differential equation

A B S T R A C T

Convective heat transfer due to laminar fluid flow past a flat plate is a standard problem in heat transfer. While constant heat flux or temperature along the plate is often assumed for solving such problems, there may be several practical scenarios where the heat flux along the plate varies as a function of both space and time. Developing an analytical solution for the resulting plate temperature distribution is important for understanding and optimizing the thermal performance of such systems. While some work exists on analyzing problems with time-dependent or space-dependent heat flux, there is a lack of work on the general problem where the heat flux is a function of both space and time. This paper presents a solution for this problem by solving the integral form of the energy equation, along with the use of fourth-order Karman-Pohlhausen polynomials for velocity and temperature distributions in the momentum and thermal boundary layers. A non-linear, first order, hyperbolic partial differential equation for the plate temperature is derived in response to the time- and space-varying plate heat flux. This equation is shown to agree well with results from past work for several special cases. Numerical solutions for the generalized equation are presented. Based on this approach, the plate temperature distribution is predicted for several heat flux profiles that may be of interest in practical applications. Results from this work improve our understanding of unsteady convective heat transfer, and contribute towards modeling and optimization of practical engineering systems where such phenomena occur.

1. Introduction

Heat transfer between a flat plate and fluid flow is a classical problem in convective heat transfer [1–3], with applications in a wide variety of engineering applications. As the fluid flows past the plate, hydrodynamic and thermal boundary layers develop, across which, momentum and thermal transport occurs. Analytical solutions of several such problems are available in textbooks [1–3] and in the research literature. Despite its classical, long-standing nature, external convective heat transfer in flat plate boundary layers continues to be a research topic of current interest [4,5].

Several simplifying assumptions are often made in order to solve the underlying momentum and energy conservation equations, including steady state, uniform freestream velocity and temperature, laminar, incompressible flow, and uniform plate temperature or heat flux [1]. Two distinct approaches have been used for analytically solving such problems. In the differential approach, the underlying momentum and energy conservation equations, written in differential form are solved in order to determine the velocity and temperature distributions [1]. Self-similar solutions are often sought for these equations [2], although this approach works only for the simplest problems, such as a flat plate with constant temperature. In most other problems, one must rely on numerical computation using finite-volume or similar methods. On the other hand, integral approaches [1] seek a solution that satisfies the energy equation integrated over the boundary layer, which results in some error, but also, considerable simplification. In the Karman-Pohlhausen integral approach [1], polynomial forms of velocity and temperature profiles in the boundary layers are written in order to satisfy various boundary conditions at the plate and edge of the boundary layer. These polynomials are then used in the integral energy balance equation to simplify and solve the problem. The assumption of polynomial forms for velocity and temperature profiles is an approximation, but has been justified because parameters governing plate-flow interactions such as the Nusselt number depend only on processes near the plate and are relatively unaffected by the velocity/temperature distribution in the remainder of the boundary layer [1,6,7]. Polynomial expressions up to the fourth order have been used in the past [6,7].

While flow past a flat plate at a constant temperature or flux are amongst the simplest problems in external convective heat

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https://doi.org/10.1016/j.ijheatmasstransfer.2021.121084
0017-9310/

Nomenclature

c	specific heat capacity, $Jkg^{-1}K^{-1}$
C	coefficient, $C = 5.83\sqrt{\frac{\nu}{U_\infty}}$, $m^{-0.5}$
k	thermal conductivity, $Wm^{-1}K^{-1}$
Nu	Nusselt number
Pr	Prandtl number
q	heat flux, Wm^{-2}
Re	Reynolds number
t	time, s
T	temperature, K
U,V	velocities in x and y directions, respectively, ms^{-1}
x,y	spatial coordinate, m
θ	relative temperature, K
α	thermal diffusivity, m^2s^{-1}
ν	kinematic viscosity, m^2s^{-1}
ρ	mass density kgm^{-3}
δ	boundary layer thickness, m

Subscripts

p	plate
t	thermal
∞	freestream

transfer, analytical solutions for several other, more complicated problems have also been reported. For example, spatial variation in plate temperature or heat flux has been accounted for using superimposition or Duhamel's theorem [8]. The Karman-Pohlhausen approach has also been used for solving this problem [6]. The steady-state problem with a specific heat flux profile has been solved using a differential approach [9]. While this results in an exact solution, it is valid only for the specific flux profile assumed, and is not valid in general. A discrete Green's function approach has also been developed to solve such problems [4,10]. Solutions for several other steady-state problems have also been summarized [8]. Unsteady convective heat transfer has also been analyzed [11,12]. For example, the effect of time-varying plate flux, including a step change in heat flux has been accounted for using the Karman-Pohlhausen approach [7], resulting in a partial differential equation for the plate temperature as a function of time and space. This problem has also been solved using Laplace transform approach [13] and Green's functions [14]. Other considerations such as turbulence [15], non-flat surface [16] and finite thickness plate [17] have also been investigated. Unsteady convective heat transfer analysis often results in a differential equation that must be solved numerically due to the lack of a closed-form analytical solution.

While there is, in general, extensive literature on external convective heat transfer, one particular problem that has not been discussed much in the literature is that of a flat plate with heat flux that varies both in space and in time, i.e. $q_p = q_p(x,t)$. While solutions have been presented for problems where the heat flux varies only in space, $q_p(x)$ [6] or only in time, $q_p(t)$ [7], a solution of the general $q_p(x,t)$ problem is desirable for many practical applications. For example, the problem of fluid flow over a bed of phase change material (PCM) occurs commonly in latent energy storage systems [18,19]. In such a case, heat flux at the fluid-PCM interface varies over space due to boundary layer development in the fluid flow, and also varies in time because of phase change front propagation into the PCM over time [20]. While such problems involving flow over a PCM have been solved by assuming a specific form of the Nusselt number [21,22], this assumption is likely to be inaccurate due to the expected time and spatial variation. Another example is the cooling of a Li-ion cell [23]. Thermal management of Li-ion cells – used commonly for energy storage and conversion in electric vehicles and other applications – often involves flow of a

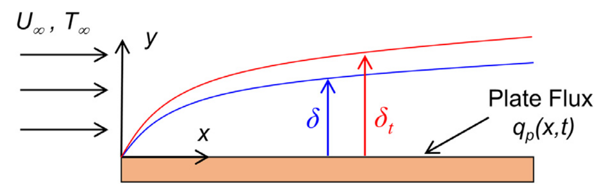

Fig. 1. Schematic of the geometry considered in this work, comprising laminar, incompressible flow past a thin flat plate with time- and space-varying heat flux.

coolant fluid over the cell or battery pack. In such a case, the interface flux may vary with space due to boundary layer growth, and may also vary in time due to fluctuations in heat generation inside the cell in response to transient changes in the electrical load. Analysis of convective heat transfer from a plate with heat flux that varies in both space and time is critical for understanding and optimizing these and other related engineering applications. Unlike problems with only time-varying or only spatially-varying heat flux, the more general problem identified above has not been sufficiently addressed in the literature.

This paper presents a solution for external convective heat transfer between a flat plate and fluid flow where the plate heat flux changes with time as well as space. Laminar incompressible flow with constant freestream velocity and temperature is assumed. Velocity and temperature distributions are represented by fourth-order Karman-Pohlhausen polynomials. It is shown that the plate temperature is governed by a first-order hyperbolic partial differential equation involving the given heat flux and its derivatives. While an analytical solution for this equation is unlikely, the equation is integrated numerically to determine the plate temperature as a function of space and time for any arbitrary heat flux distribution. Results from this work are shown to agree well with past results for special cases of time-varying [7], spatially-varying [6] or constant flux [1]. Various general cases of time- and space-varying heat flux, such as those that might arise in applications discussed above are analyzed.

2. Mathematical modeling

As shown schematically in Fig. 1, consider laminar, incompressible fluid flow past a thin, one-dimensional flat plate in which the plate flux, q_p, varies both in space and in time. Uniform freestream velocity U_∞ and temperature T_∞ are assumed. All properties are assumed to be independent of temperature. Viscous dissipation is neglected. The interest here is to determine the nature of heat transfer between the flat plate and the fluid flow. In particular, the resulting plate temperature $T_p(x,t)$ due to the spatial and time variation in flux, $q_p(x,t)$, is of interest.

Under the assumptions listed above, and referring to Fig. 1 for the coordinate system, the governing equations for conservation of mass, momentum and energy can be written as

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \quad (1)$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \nu\frac{\partial^2 U}{\partial y^2} \quad (2)$$

$$\alpha\frac{\partial^2 \theta}{\partial y^2} = \frac{\partial \theta}{\partial t} + U\frac{\partial \theta}{\partial x} + V\frac{\partial \theta}{\partial y} \quad (3)$$

where $\theta = T - T_\infty$.

The boundary conditions are

$$U = U_\infty; \ \ \theta = 0 \ at \ x = 0 \quad (4)$$

$$U = 0; V = 0 \ at \ y = 0 \quad (5)$$

$$U = U_\infty \ at \ y \geq \delta \tag{6}$$

$$\theta = 0 \ at \ y \geq \delta_t \tag{7}$$

$$-k\frac{\partial \theta}{\partial y} = q_p(x,t) \ at \ y = 0 \tag{8}$$

Eq. (4) represents the uniform freestream velocity and temperature of the fluid flow. Eq. (5) is the no slip condition at the plate. Eqs. (6) and (7) are based on δ and δ_t, the momentum and thermal boundary layer thicknesses, respectively. Eq. (8) is the given heat flux at the flat plate. In addition, the initial temperature field, $\theta\ (x,y,t=0)$, is assumed to be zero. The interest is in solving this set of equations to determine the plate temperature $\theta_p(x,t) = \theta\ (x,y=0,t)$.

2.1. Solution procedure

It is assumed that hydrodynamic boundary layer growth is unaffected by heat transfer. Integral analysis of momentum transfer in this problem can be shown to result in the following expression for the momentum boundary layer thickness, δ [6,7]

$$\delta(x) = 5.83\sqrt{\frac{\nu x}{U_\infty}} \tag{9}$$

In order to solve this problem, the velocity and temperature distributions in their respective boundary layers are assumed to be given by fourth-order Karman-Pohlhausen polynomials [1,6,7] as follows:

$$\frac{U(x,y,t)}{U_\infty} = 2\frac{y}{\delta} - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4 \tag{10}$$

$$\frac{\theta(x,y,t)}{\theta_p} = 1 - 2\frac{y}{\delta_t} + 2\left(\frac{y}{\delta_t}\right)^3 - \left(\frac{y}{\delta_t}\right)^4 \tag{11}$$

These polynomials that already satisfy several boundary conditions in the problem are substituted into the integrated form of the energy equation in order to satisfy overall energy conservation. To do so, Eq. (3) is first integrated from y=0 to y=δ_t, resulting in

$$\frac{\partial}{\partial t}\int_0^{\delta_t}\theta dy + \frac{\partial}{\partial x}\int_0^{\delta_t}U\theta dy = -\alpha\left(\frac{\partial\theta}{\partial y}\right)_{y=0} = \frac{q_p}{\rho c} \tag{12}$$

By substituting the polynomial form of θ from Eq. (11) into the first term of Eq. (12), it can be shown that

$$\frac{\partial}{\partial t}\int_0^{\delta_t}\theta dy = \frac{6}{5}\frac{k\theta_p}{q_p}\frac{\partial\theta_p}{\partial t} - \frac{3}{5}\frac{k\theta_p^2}{q_p^2}\frac{\partial q_p}{\partial t} \tag{13}$$

Also, substituting Eqs. (10) and (11) into the second term in Eq. (12) results in

$$\frac{\partial}{\partial x}\int_0^{\delta_t}U\theta dy = U_\infty\frac{\partial}{\partial x}\left[\frac{8}{15}\frac{k^2\theta_p^3}{q_p^2C\sqrt{x}} - \frac{12}{35}\frac{k^4\theta_p^5}{q_p^4C^3x\sqrt{x}} + \frac{8}{45}\frac{k^5\theta_p^6}{q_p^5C^4x^2}\right] \tag{14}$$

where $C = 5.83\sqrt{\frac{\nu}{U_\infty}}$.

Recognizing that θ_p and q_p are both functions of x and t, Eq. (14) can be differentiated by parts to result in

$$\frac{\partial}{\partial x}\int_0^{\delta_t}U\theta dy = U_\infty\left[\begin{array}{c}\frac{8}{15}\frac{k^2}{C}\left(-\frac{2}{q_p^3}\frac{\partial q_p}{\partial x}\frac{\theta_p^3}{\sqrt{x}} + \frac{1}{q_p^2}\left(\frac{3\theta_p^2}{\sqrt{x}}\frac{\partial\theta_p}{\partial x} - \frac{\theta_p^3}{2x\sqrt{x}}\right)\right)\\ -\frac{12}{35}\frac{k^4}{C^3}\left(-\frac{4}{q_p^5}\frac{\partial q_p}{\partial x}\frac{\theta_p^5}{x\sqrt{x}} + \frac{1}{q_p^4}\left(\frac{5\theta_p^4}{x\sqrt{x}}\frac{\partial\theta_p}{\partial x} - \frac{3\theta_p^5}{2x^2\sqrt{x}}\right)\right)\\ +\frac{8}{45}\frac{k^5}{C^4}\left(-\frac{5}{q_p^6}\frac{\partial q_p}{\partial x}\frac{\theta_p^6}{x^2} + \frac{1}{q_p^5}\left(\frac{6\theta_p^5}{x^2}\frac{\partial\theta_p}{\partial x} - \frac{2\theta_p^6}{x^3}\right)\right)\end{array}\right] \tag{15}$$

Substituting Eqs. (13) and (15) in the integral energy equation given in Eq. (12), followed by mathematical simplification results in a partial differential equation of the form

$$A(\theta_p,x,t)\frac{\partial\theta_p}{\partial t} + B(\theta_p,x,t)\frac{\partial\theta_p}{\partial x} = F(\theta_p,x,t) \tag{16}$$

where

$$A(\theta_p,x,t) = \frac{6}{5}\frac{\theta_p q_p^4 C}{k^3} \tag{17}$$

$$B(\theta_p,x,t) = U_\infty\left[\frac{8}{5}\frac{\theta_p^2q_p^3}{k^2\sqrt{x}} - \frac{12}{7}\frac{\theta_p^4q_p}{C^2x\sqrt{x}} + \frac{16}{15}\frac{\theta_p^5}{C^3x^2}k\right] \tag{18}$$

$$\begin{aligned}F(\theta_p,x,t) = &\frac{\alpha q_p^6C}{k^5} + \frac{3C\theta_p^2q_p^3}{5k^3}\frac{\partial q_p}{\partial t} + \left[\frac{4\theta_p^3q_p^3}{15k^2x\sqrt{x}} - \frac{18q_p\theta_p^5}{35C^2x^2\sqrt{x}} + \frac{16k\theta_p^6}{45C^3x^3}\right]U_\infty\\ &-\frac{\partial q_p}{\partial x}\left[-\frac{16q_p^2\theta_p^3}{15k^2\sqrt{x}} + \frac{48\theta_p^5}{35C^2x\sqrt{x}} - \frac{8k\theta_p^6}{9C^3x^2q_p}\right]U_\infty\end{aligned} \tag{19}$$

Eq. (16) along with coefficients defined in Eqs. (17)–(19) represents the governing equation for the plate temperature for the general case considered here, where the imposed plate flux varies with both x and t. Note that the heat flux function $q_p(x,t)$ and its derivatives appearing in the coefficients of Eq. (16) are known in advance, and therefore, Eq. (16) is a non-linear, first-order hyperbolic partial differential equation in $\theta_p(x,t)$. While it is unlikely that Eq. (16) has a general closed-form solution, hyperbolic differential equations such as Eq. (16) can be solved numerically.

2.2. Numerical integration of Eq. (16)

In order to numerically compute the solution of Eqs. (16)–(19), discretization in x direction is carried out to convert the partial differential equation into a coupled system of ordinary differential equations in $\theta_{p,i}(t)$ where the subscript i refers to the discretized spatial location. The coupled system of ordinary differential equations is then solved by numerical integration over time with a three-stage, third-order Runge-Kutta solver using adaptive timestepping [24]. The initial condition provides the initial state needed for starting the integration process.

3. Results and discussion

3.1. Special cases

It is instructive to compare special cases of the general results derived in Section 2 with past work that investigated specific special cases of this problem.

In case the plate flux is a function of time alone, i.e. q_p=$q_p(t)$, then the last set of terms in Eq. (19) that appear with $\frac{\partial q_p}{\partial x}$ can be set to zero. Even if the plate flux may not be spatially dependent, the plate temperature will still be a function of both x and t due to boundary layer development. For this case, the general governing equation can be simplified to

$$\begin{aligned}&\frac{6}{5}\frac{\theta_pq_p^4C}{k^3}\frac{\partial\theta_p}{\partial t} + \left[\frac{8}{5}\frac{\theta_p^2q_p^3}{k^2\sqrt{x}} - \frac{12}{7}\frac{\theta_p^4q_p}{C^2x\sqrt{x}} + \frac{16}{15}\frac{\theta_p^5}{C^3x^2}k\right]U_\infty\frac{\partial\theta_p}{\partial x}\\ &-\frac{3C\theta_p^2q_p^3}{5k^3}\frac{\partial q_p}{\partial t} - \left[\frac{4\theta_p^3q_p^3}{15k^2x\sqrt{x}} - \frac{18q_p\theta_p^5}{35C^2x^2\sqrt{x}} + \frac{16k\theta_p^6}{45C^3x^3}\right]U_\infty = \frac{\alpha q_p^6C}{k^5}\end{aligned} \tag{20}$$

Eq. (20) matches exactly with the result from Lachi, et al. [7] that considered the specific case of time-varying heat flux on the flat plate.

Further, the case of only spatially-varying plate flux, i.e. $q_p = q_p(x)$ is of interest. In such a case, the $\frac{3C\theta_p^2q_p^3}{5k^3}\frac{\partial q_p}{\partial t}$ term appearing in Eq. (19) can be eliminated. Therefore, the governing equation for this case is

$$\frac{6}{5}\frac{\theta_p q_p^4 C}{k^3}\frac{\partial\theta_p}{\partial t}+\left[\frac{8}{5}\frac{\theta_p^2 q_p^3}{k^2\sqrt{x}}-\frac{12}{7}\frac{\theta_p^4 q_p}{C^2 x\sqrt{x}}+\frac{16}{15}\frac{\theta_p^5}{C^3x^2}k\right]U_\infty\frac{\partial\theta_p}{\partial x}$$
$$+\frac{\partial q_p}{\partial x}\left[-\frac{16q_p^2\theta_p^3}{15k^2\sqrt{x}}+\frac{48\theta_p^5}{35C^2x\sqrt{x}}-\frac{8k\theta_p^6}{9C^3x^2q_p}\right]U_\infty$$
$$+\left[-\frac{4\theta_p^3q_p^3}{15k^2x\sqrt{x}}+\frac{18q_p\theta_p^5}{35C^2x^2\sqrt{x}}-\frac{16k\theta_p^6}{45C^3x^3}\right]U_\infty=\frac{\alpha q_p^6 C}{k^5} \quad (21)$$

The transient problem of spatially-varying plate flux was presented by Polidori and Padet [6]. Appendix A shows that the governing equation obtained above by simplifying the equation for the general $q_p(x,t)$ case agrees exactly with results from Polidori and Padet [6].

Further, the steady state component of the spatially-varying heat flux problem is a standard problem, for which, the solution is available [1] as follows:

$$\theta_p(x)=\frac{0.623}{k}Pr^{-1/3}Re_x^{-1/2}\int_0^x q_p(\xi)\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{-2/3}d\xi \quad (22)$$

Where $Re_x=\frac{U_\infty x}{\nu}$ is the Reynolds number. In comparison, in the present work, for steady state conditions with $q_p=q_p(x)$, Eq. (21) can be further simplified to

$$\left[\frac{8}{5}\frac{\theta_p^2 q_p^3}{k^2\sqrt{x}}-\frac{12}{7}\frac{\theta_p^4 q_p}{C^2 x\sqrt{x}}+\frac{16}{15}\frac{\theta_p^5}{C^3x^2}k\right]U_\infty\frac{\partial\theta_p}{\partial x}$$
$$+\frac{\partial q_p}{\partial x}\left[-\frac{16q_p^2\theta_p^3}{15k^2\sqrt{x}}+\frac{48\theta_p^5}{35C^2x\sqrt{x}}-\frac{8k\theta_p^6}{9C^3x^2q_p}\right]U_\infty$$
$$+\left[-\frac{4\theta_p^3q_p^3}{15k^2x\sqrt{x}}+\frac{18q_p\theta_p^5}{35C^2x^2\sqrt{x}}-\frac{16k\theta_p^6}{45C^3x^3}\right]U_\infty=\frac{\alpha q_p^6 C}{k^5} \quad (23)$$

Eq. (23) is an ordinary differential equation in $\theta_p(x)$, which is difficult to solve analytically due to its non-linear nature. However, a numerical computation of Eq. (23) can be compared against Eq. (22).

Finally, note that the solution for a flat plate with constant heat flux is [1]

$$\theta_p(x)=\frac{q_p}{0.453k}Pr^{-1/3}Re_x^{-1/2}x \quad (24)$$

whereas, the present work, through elimination of terms appearing with $\frac{\partial q_p}{\partial x}$ in Eq. (23) results in

$$\left[\frac{8}{5}\frac{\theta_p^2 q_p^3}{k^2\sqrt{x}}-\frac{12}{7}\frac{\theta_p^4 q_p}{C^2 x\sqrt{x}}+\frac{16}{15}\frac{\theta_p^5}{C^3x^2}k\right]U_\infty\frac{\partial\theta_p}{\partial x}$$
$$+\left[-\frac{4\theta_p^3q_p^3}{15k^2x\sqrt{x}}+\frac{18q_p\theta_p^5}{35C^2x^2\sqrt{x}}-\frac{16k\theta_p^6}{45C^3x^3}\right]U_\infty=\frac{\alpha q_p^6 C}{k^5} \quad (25)$$

The equations for special cases resulting from the present work are plotted along with past results in the next section.

3.2. Model validation

The analytical model derived in this work is compared against results from past papers that have presented theoretical analysis of similar problems. Firstly, results are compared with Lachi, et al. [7], who presented the analysis of unsteady convective heat transfer with a time-dependent flat plate heat flux, $q_p(t)$. For this comparison, a specific case of step change heat flux considered by Lachi, et al. is also implemented in the present analytical model. This step change involves a change of heat flux from φ_1 to φ_2 ($\varphi_2 > \varphi_1$) at time t_0. Eq. (20) is solved for this specific heat flux profile in order to compute the plate temperature as a function of x and t. Under the same freestream conditions and fluid properties, a comparison with Lachi, et al. [7] is presented in Fig. 2, where plate temperature and Nusselt number at multiple locations are plotted as functions of time in Fig. 2(a) and (b), respectively. In general, there is very good agreement between the present work and Lachi, et al. for the special case of time-dependent plate heat flux. The plate temperature rises with time, including a sharp rise beyond $t=t_0$ when the heat flux undergoes a step change. Further, the greater the value of x, the higher is the temperature, which is due to boundary layer growth and diminished heat transfer from the plate at large x. Similarly, Nu reduces with time and then undergoes a large increase at $t=t_0$ due to the increased heat flux, as expected, and then finally reduces with increasing time. The larger the value of x, the higher is the value of Nu, which is also expected.

Further, the present work is compared with Polidori and Padet [6] for the special case of spatially-varying heat flux, $q_p(x)$. A specific form of $q_p(x)=150\cdot\exp(-10x)W/m^2$ used in their work is also implemented in the present model, Eq. (21), which pertains to the special case of spatially-varying heat flux, $q_p(x)$. Comparison of the present work with Polidori and Padet is presented in Fig. 3, which plots the spatial distribution in the plate temperature at multiple times prior to steady state. At each time, the plate temperature increases with x, and then decreases after reaching a maxima. This is because the plate flux reduces exponentially as x increases. At each time, there is good agreement between the past work and the present model. Please note that the approach for numerical solution of the derived equations is different between Polidori and Padet and the present work. Polidori and Padet adopted an explicit finite-difference scheme [6], whereas in the present work, the partial differential equation is discretized in space, and the resulting system of ordinary differential equations is solved numerically using a three-stage, third-order, Runge-Kutta method. This difference between the two approaches may explain the small discrepancy between the two sets of curves in Fig. 3.

The problem of spatially-varying plate temperature has also been solved in steady state using superposition methods [1]. In short, the solution for a problem with constant plate temperature and an nunheated length has been derived and then superimposed based on linearity of the problem to determine the solution for the spatially-varying plate temperature problem. A comparison of the present work with these results is presented in Fig. 4(a) and (b) for two specific $q_p(x)$ profiles – linear, and one in which $q_p(x)$ reduces proportional to $\sqrt{x}$. In both cases, there is good agreement between the two. Note that this good agreement is particularly encouraging because the approach in the superposition-based method is distinctly different from the approach in the present work.

Finally, comparison with past work is carried out for the simplest case of a constant heat flux plate. A solution for this case has been presented in Kays and Crawford [1], and is reproduced as Eq. (24) in the present work. Fig. 5 presents the variation in Nu as a function of x for a constant heat flux plate, based on the present model as well as past results. The two are in excellent agreement with each other.

The good agreement between the present work and various past results for special cases of $q_p(t)$, $q_p(x)$ and constant q_p provides confidence in the present approach.

3.3. Plate temperature for specific $q_p(x,t)$ functions

The plate temperature distribution is determined for several representative plate flux profiles in order to further demonstrate the capability of the present analytical model.

In order to analyze a scenario where the plate flux changes with both time and space, a plate flux profile given by $q_p(x,t)=A+Bx+Ct$ is considered, where $A=200\ W/m^2$, $B=-400\ (\frac{W}{m^2})/m$ and $C=-50\ (\frac{W}{m^2})/s$. The freestream velocity is assumed to be 1 m/s. Room temperature properties of air are used. The resulting plate temperature distribution determined by solving Eq. (16) is plotted in Fig. 6. Plate temperature as a function of x at three different times is plotted in Fig. 6(a), while plate temperature as a

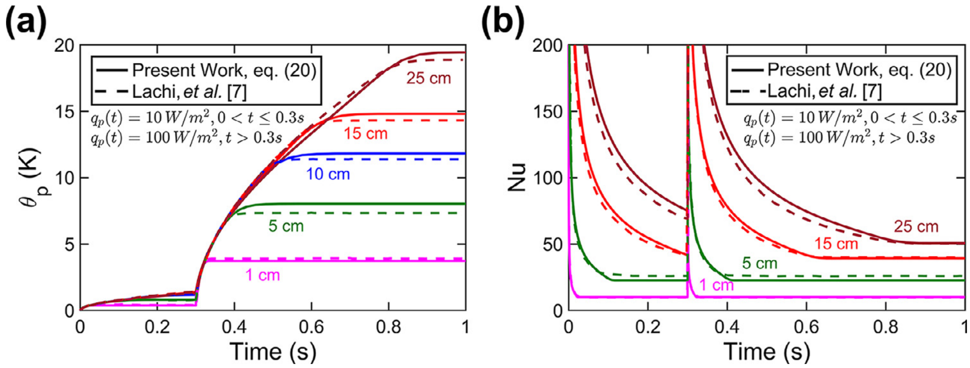

Fig. 2. Comparison of present work with Lachi, et al. [7] for the special case of time-dependent heat flux, $q_p(t) = \begin{cases} 10\,W/m^2, & 0 < t \le 0.3\,s \\ 100,\ W/m^2\, t > 0.3\,s \end{cases}$: (a) Plate temperature, θ_p vs t at multiple x, and (b) Nu vs t at the plate at multiple x.

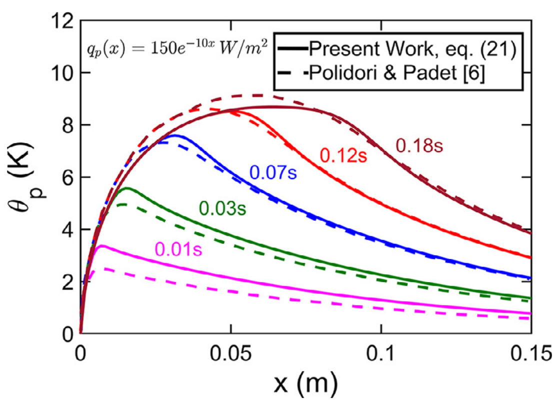

Fig. 3. Comparison of present work with Polidori and Padet [6] for the special case of spatially-varying heat flux: Plate temperature, θ_p as a function of x at multiple different times for $q_p(x) = 150e^{-10x}\,W/m^2$. Note that x is in m.

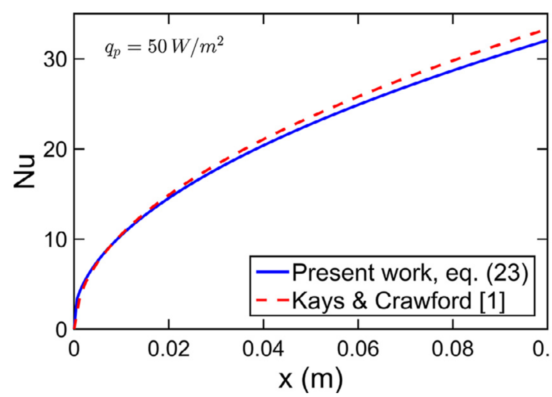

Fig. 5. Comparison of present work with Kays and Crawford [1] for the special case of steady state constant heat flux: Nu as a function of x.

function of time at four different locations is plotted in Fig. 6(b). Fig. 6(a) shows, as expected, that the plate temperature increases with x at any given time. A saturation effect at large values of x is also seen, as expected, due to the saturation in boundary layer growth. The plate temperature decreases with time due to the decreasing nature of q_p with time. Fig. 6(b) shows that the plate temperature at any specific location increases sharply first, reaches a peak and then slowly decreases. The initial rise in temperature is due to heating up of the plate, which is followed by a cool down because the plate flux reduces with time according to the assumed form of the plate flux distribution, while the plate continues to be convectively cooled by the flow. Fig. 6(b) shows that the larger the value of x, the larger is the temperature rise, which is expected due to the increasing boundary layer thickness and therefore, diminishing convective cooling as x increases.

Fig. 7 presents results for a similar heat flux distribution, given by $q_p(x,t) = A + B\sqrt{x} + C\sqrt{t}$, with $A = 500\ W/m^2$, $B =$

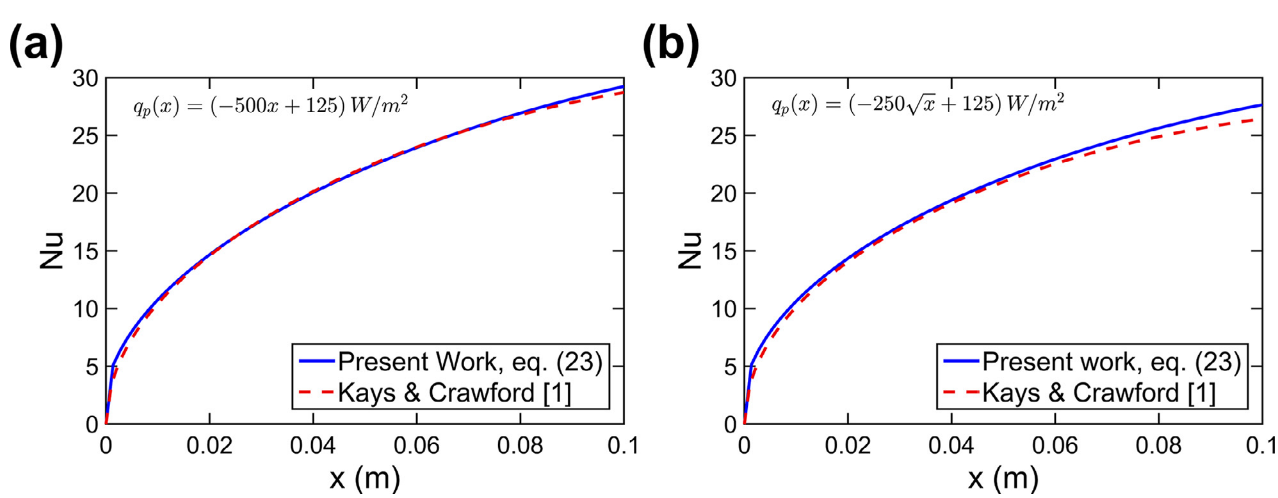

Fig. 4. Comparison of present work with Kays and Crawford [1] for the special case of steady state spatially-dependent heat flux: (a) plate temperature in the form of Nusselt number, Nu, as a function of x for (a) linear $q_p(x) = -500x + 125\,W/m^2$, (b) non-linear $q_p(x) = -250\sqrt{x} + 125\,W/m^2$. Note that x is in m.

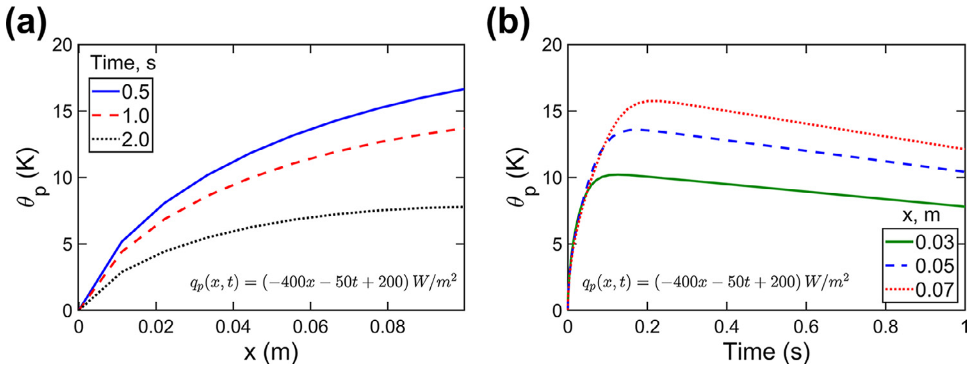

Fig. 6. Plate temperature, θ_p as a function of (a) x at multiple times, and (b) time at multiple x for a general heat flux distribution given by $q_p(x,t) = [200 - 400x - 50t]\,W/m^2$.

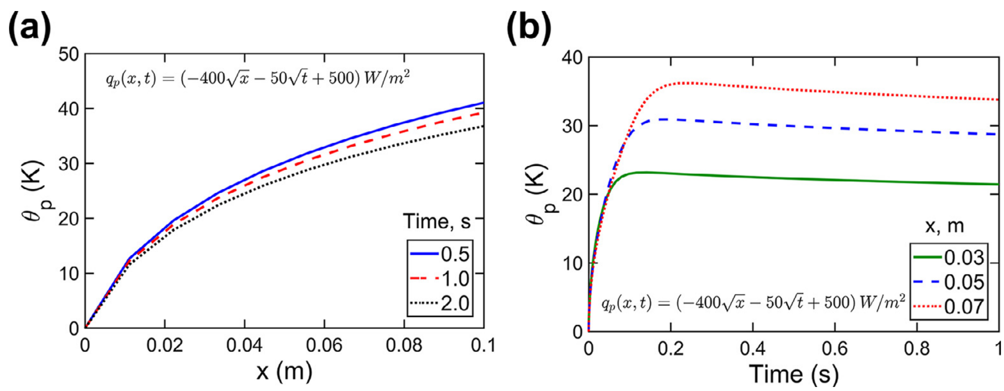

Fig. 7. Plate temperature, θ_p as a function of (a) x at multiple times, and (b) time at multiple x for a general heat flux distribution given by $q_p(x,t) = [500 - 400\sqrt{x} - 50\sqrt{t}]\,W/m^2$.

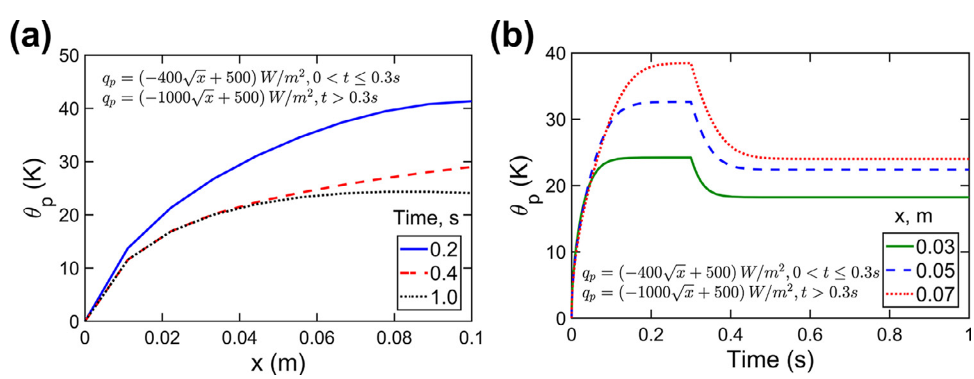

Fig. 8. Plate temperature, θ_p as a function of (a) x at multiple times, and (b) time at multiple x for a general heat flux distribution given by $q_p(x,t) = [500 - 400\sqrt{x}]\,W/m^2$ for $t < 0.3s$ and $q_p(x,t) = [500 - 1000\sqrt{x}]\ W/m^2$ afterwards.

$-400\ (\frac{W}{m^2})/\sqrt{m}$ and $C = -50\ (\frac{W}{m^2})/\sqrt{s}$. Here, the dependence of plate flux of x and t is weaker than in the previous case. Other problem parameters such as freestream velocity as the same as the previous Figure. Fig. 7(a) and (b) plot the variation in plate temperature as a function of x and time, respectively. Similar to Fig. 6(a), the plate temperature distribution in Fig. 7(a) shows increasing temperature with x at any time, which is consistent with reduced convective cooling at large x. Compared to Fig. 6(a), the three curves in Fig. 7(a) at three different times are much closer to each other, which is likely due to the slower rate of reduction in the plate heat flux with time compared to Fig. 6. Note that the assumed plate flux for results shown in Fig. 7 decays as $\sqrt{x}$ and $\sqrt{t}$, compared to the linear decay for Fig. 6. Similar to Fig. 6(b), there is initial rise in plate temperature at any given location, as shown in Fig. 7(b). This is followed by a reduction in temperature, but at a much slower rate than in the previous case. This is also likely due to the weaker decay in the plate flux compared to the previous case.

Finally, a step function change in the plate flux is also considered, wherein the plate flux is given by $q_p = A + B_1\sqrt{x}$ for $0<t<0.3$ s, and $q_p = A + B_2\sqrt{x}$ afterwards. The numerical values of these parameters are $A = 500\frac{W}{m^2}$, $B_1 = -400\ (\frac{W}{m^2})/\sqrt{m}$ and $B_2 = -1000\ (\frac{W}{m^2})/\sqrt{m}$. This constitutes a sharp drop in the plate flux after t=0.3 s. The resulting plate temperature distribution is plotted in Fig. 8. Spatial distribution in the plate temperature at three different times is plotted in Fig. 8(a), while plate temperature as a function of time at three different locations is plotted in Fig. 8(b). As expected, the plate temperature at any given location increases first with time up to t=0.3 s, with the temperature being greater at large values of x. This is followed by a gradual reduction due to the step change in the flux distribution. Eventually, the plate temperature at each location reaches a steady state. Temperature at steady state increases with increasing value of x, which is due to reduced convective cooling at large x. The plate temperature distribution at three different times, shown in Fig. 8(a) indicates that the plate temperature rises with x and eventually plateaus out. Similar to Fig. 8(b), this occurs due to the reduced convective cooling at large x, combined with the gradual reduction in the plate flux with x, based on the assumed flux distribution.

Figs. 6–8 demonstrate the capability of the theoretical model derived in this work to account for a given plate heat flux as a function of both space and time in order to predict the resulting temperature distribution. Experimental data on the plate heat flux, if available, may also be inserted into Eqs. (16)–(19) to predict the temperature distribution. In such a case, the derivatives $\frac{\partial q_p}{\partial t}$ and $\frac{\partial q_p}{\partial x}$ appearing in Eq. (19) may need to be evaluated numerically.

4. Conclusions

Understanding the nature of temperature distribution on a flat plate due to convective cooling in response to a plate heat flux that varies in both space and time is important for design and optimization of several practical engineering systems such as latent heat energy storage systems and thermal management of Li-ion batteries. In the past, such problems have been solved when the plate flux is a function of only time or only x. The present work generalizes this by considering the plate heat flux to be a function of both space and time. The generalized solution derived in this work is based on the integral approach and uses fourth-order Karman-Pohlhausen polynomials, which have been shown to offer reasonable accuracy. Results are shown to agree well with past work for specific cases of the general problem discussed in this work.

The present work does not account for second-order effects such as thermal resistance and capacitance of the flat plate, which may require solving a conjugate heat transfer problem, for which, the present results may be helpful. Also, turbulent effects, which may be important in specific applications are not accounted for in the present work.

The present work improves our fundamental understanding of external convective heat transfer. The results derived here may contribute towards design and optimization of practical engineering systems.

Declaration of Competing Interest

None

CRediT authorship contribution statement

Amirhossein Mostafavi: Methodology, Investigation, Visualization, Data curtion, Writing – original draft, Writing – review & editing. **Ankur Jain:** Conceptualization, Methodology, Supervision, Project administration, Writing – original draft, Writing – review & editing.

Acknowledgment

This material is based upon work supported by CAREER Award No. CBET-1554183 from the National Science Foundation.

Appendix A. Proof that results from present work agree with Polidori and Padet [6] for special case of $q_p(x)$

The general equation derived in the present work for the plate temperature due to a time- and spatially-varying heat flux $q_p(x,t)$ (Eq. (16)) reduces to a simpler form, Eq. (21), for the special case of a heat flux that varies only in x, i.e., $q_p(x)$. This specific problem has been discussed in the past by Polidori and Padet [6]. In order to establish that Eq. (21) matches with the results in Polidori and Padet [6], one may begin with Eq. (4) of their paper. This equation, with variable names changed to match the present work is

$$\frac{3}{10}\frac{\partial}{\partial t}[\theta_p\delta_t] + U_\infty\frac{\partial}{\partial x}\left[\theta_p\left(\frac{2}{15}\frac{\delta_t^2}{\delta} - \frac{3}{140}\frac{\delta_t^4}{\delta^3} + \frac{1}{180}\frac{\delta_t^5}{\delta^4}\right)\right] = \frac{\nu q_p}{k\cdot Pr} \tag{A.1}$$

Introducing $\delta(x) = C\sqrt{x}$ for the momentum boundary layer thickness and $\delta_t = \frac{2k\theta_p}{q_p}$, one may simplify Eq. (A.1) to

$$\frac{6}{5}\frac{k\theta_p}{q_p}\frac{\partial\theta_p}{\partial t} + U_\infty\frac{\partial}{\partial x}\left[\left(\frac{8}{15}\frac{k^2\theta_p^3}{Cq_p^2\sqrt{x}} - \frac{12}{35}\frac{k^4\theta_p^5}{C^3q_p^4x\sqrt{x}} + \frac{8}{45}\frac{k^5\theta_p^6}{C^4q_p^5x^2}\right)\right] = \frac{\alpha q_p}{k} \tag{A.2}$$

Since $\theta_p$and q_p are both functions of x, the second term in Eq. (A.2) is differentiated by parts. In addition, the entire equation is multiplied by $\frac{Cq_p^5}{k^4}$, resulting in

$$\begin{aligned}\frac{6}{5}\frac{\theta_p q_p^4 C}{k^3}\frac{\partial\theta_p}{\partial t} &+ \left[\frac{8}{5}\frac{\theta_p^2 q_p^3}{k^2\sqrt{x}} - \frac{12}{7}\frac{\theta_p^4 q_p}{C^2x\sqrt{x}} + \frac{16}{15}\frac{\theta_p^5}{C^3x^2}k\right]U_\infty\frac{\partial\theta_p}{\partial x}\\ &+ \frac{\partial q_p}{\partial x}\left[-\frac{16q_p^2\theta_p^3}{15k^2\sqrt{x}} + \frac{48\theta_p^5}{35C^2x\sqrt{x}} - \frac{8k\theta_p^6}{9C^3x^2q_p}\right]U_\infty\\ &+ \left[-\frac{4\theta_p^3q_p^3}{15k^2x\sqrt{x}} + \frac{18q_p\theta_p^5}{35C^2x^2\sqrt{x}} - \frac{16k\theta_p^6}{45C^3x^3}\right]U_\infty = \frac{\alpha q_p^6 C}{k^5}\end{aligned} \tag{A.3}$$

Eq. (A.3) is identical to the result from the present work for the special case of $q_p(x)$, given by Eq. (21). Therefore, the generalized result derived in the present work reduces to the one presented by Polidori and Padet [6] for the special case of a plate heat flux that varies only with x.

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